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Asymptotics of Kolmogorov Diameters for Some Classes of Harmonic Functions on Spheroids

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Let Γ_D^K be the unit ball of the space of all bounded harmonic functions in a domain D in $\mathbb{R}^3$, considered as a compact subset of the Banach space $C(K)$, where K is a compact subset of D. The old problem about the exact asymptotics for Kolmogorov diameters (widths) of this set,

$$\ln d_k(\Gamma_D^K) \sim -\tau k^{1/2}, \qquad k \to \infty,$$

is solved positively in the case when K and D are closed and open confocal spheroids, respectively (i.e., prolate or oblate ellipsoids of revolution). Using some special asymptotic formulas for the associated Legendre functions $P_n^m(\cosh \sigma)$ as $n \to \infty$ and $m/n \to \gamma \in [0, 1]$ (considered earlier by the second author), we show that the constant τ is some averaged characteristic of the pair of spheroids, expressed by means of a certain function of the variable γ, which appears within those asymptotics. Unlike the corresponding problem for analytic functions, quite well investigated, the harmonic functions case has been studied, up to now, only in the case of concentric balls. © 2000 Academic Press

1. INTRODUCTION

Let $C(K)$ be the Banach space of all continuous functions on a compact set $K \subset \mathbb{R}^n$ and let Γ_D^K be the compact subset of $C(K)$, consisting of the restrictions $x \mid K$ of all functions $x(t)$, harmonic in the open neighborhood $D \supset K$ and such that $|x(t)| \leqslant 1$ in D.

Here we study the quite old problem of the exact asymptotics of Kolmogorov diameters of those sets of harmonic functions (Mityagin and

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Tikhomirov [14]): whether a constant τ exists such that the strong asymptotic

$$\ln d_k(\Gamma_D^K) \sim -\tau k^{1/(n-1)}, \qquad k \to \infty \tag{1}$$

holds. This problem is equivalent to the problem on an exact asymptotic for ε-entropy of the set Γ_D^K; see for details Section 7.

Namely, we prove that this question is solved positively in the case when K and D are closed and open confocal spheroids, respectively (i.e., prolate or oblate ellipsoids of revolution). Using the uniform modification of the special asymptotic formulas for the associated Legendre functions $P_n^m(\cosh \sigma)$ as $n \to \infty$ and $m/n \to \gamma \in [0, 1]$ (considered first in [28]), we show that the constant τ is some averaged characteristic of the pair of spheroids, expressed by means of a certain function of the variable γ, which appears within those asymptotics.

The analogous problem for analytic functions, stated by Kolmogorov in the 50's, has been studied well [1, 5–7, 12, 15, 22–24, 29] under quite general assumptions about K and D, mainly for the one-variable case, although there are some several-variable results too (see also [10, 18, 20, 27]).

Unlike the analytic functions case, the exact asymptotic (1) has long been known only in the case of concentric balls in $\mathbb{R}^n$ ([5, 6, 14]) if $n \geqslant 3$; the case $n=2$ is more advanced (see, e.g., [16, 17, 19, 30]) due to the close relationship between harmonic and analytic functions on the plane (see Section 7).

2. PRELIMINARIES

2.1. We shall consider the one-parameter family of confocal prolate spheroids with the common focuses $(\pm 1, 0, 0)$,

$$\Phi_\sigma = \left\{ t = (t_1, t_2, t_3) \in \mathbb{R}^3 : \frac{t_1^2}{(\cosh \sigma)^2} + \frac{t_2^2 + t_3^2}{(\sinh \sigma)^2} < 1 \right\}, \qquad 0 < \sigma < \infty, \tag{2}$$

or the corresponding family of oblate spheroids,

$$\Psi_\sigma = \left\{ t = (t_1, t_2, t_3) \in \mathbb{R}^3 : \frac{t_1^2 + t_2^2}{(\cosh \sigma)^2} + \frac{t_3^2}{(\sinh \sigma)^2} < 1 \right\}, \qquad 0 < \sigma < \infty. \tag{3}$$

2.2. For an open set $\Omega \in \mathbb{R}^3$ denote by $h(\Omega)$ the space of all complex-valued harmonic functions in Ω provided with the usual topology of uniform convergence on compact subsets of Ω. For a compact set $F \subset \mathbb{R}^3$

we consider the space $h(K)$ of all harmonic germs on F with the inductive topology

$$h(F) = \lim_{s \to \infty} \operatorname{ind} h(\Omega_s),$$

where Ω_s, $s \in \mathbb{N}$, is any sequence of open neighborhoods of K, such that $\bigcap_{s \in \mathbb{N}} \Omega_s = K$ and $\Omega_{s+1} \subset \Omega_s$, $s \in \mathbb{N}$.

2.3. For a set A in a Banach space X, the k-dimensional Kolmogorov diameter (k-width) is the number ([9, 21]):

$$d_k(A) = d_k(A, X) := \inf_{L_k} \sup_{x \in A} \inf_{y \in L_k} \{ \|x - y\|_X \}, \tag{4}$$

where L_k runs through the set of all k-dimensional subspaces of X. If $Y \subset X$ is a pair of Banach spaces with linear continuous imbedding, we shall also use the notation $d_k(Y, X)$ for $d_k(A, X)$, where A is a unit ball of Y. We shall use the following notation: $hC(K)$ is the completion of $h(K)$ in the space $C(K)$; $hB(D)$ stands for the space of all bounded harmonic functions in a domain D with the uniform norm. Then for a pair of prolate spheroids we can write

$$d_k(\Gamma_{\Phi_{\sigma_1}}^{\bar{\Phi}_{\sigma_0}}) = d_k(hB(\Phi_{\sigma_1}), hC(\bar{\Phi}_{\sigma_0})). \tag{5}$$

From the definition (4) immediately follows

LEMMA 1. *For any quadruple of Banach spaces*

$$X_1 \subset X_2 \subset X_3 \subset X_4$$

with linear continuous imbeddings there exists a constant $C > 0$ *such that*

$$d_k(X_1, X_4) \leqslant C\, d_k(X_2, X_3), \qquad k \in \mathbb{N}.$$

We shall use the following well-known fact (see, e.g., [13, 21]):

PROPOSITION 2. *Let* $H_1 \subset H_0$ *be a pair of Hilbert spaces with a dense, completely continuous imbedding and let* $\{e_k\}_{k \in \mathbb{N}}$ *be their common orthogonal basis, enumerated so that the sequence*

$$\mu_k = \mu_k(H_0, H_1) := \frac{\|e_k\|_{H_1}}{\|e_k\|_{H_0}}, \qquad k \in \mathbb{N}, \tag{6}$$

is non-decreasing. Then

$$d_k(H_1, H_0) = (\mu_{k+1}(H_0, H_1))^{-1}, k \in \mathbb{N}. \tag{7}$$

2.4. The following elementary fact from geometric number theory is of measure theoretic character.

LEMMA 3. *Let $G \subset \mathbb{R}^2$ be a domain measurable in the Jordan sense (squarable), $m(G)$ its measure, and $N(tG)$ the number of all points with integer coordinates, belonging to the set $tG := \{tx : x \in G\}$, $t > 0$. Then the asymptotics*

$$N(tG) \sim m(G)\, t^2$$

holds for $t \to \infty$.

3. MAIN RESULTS

THEOREM 4. *The asymptotic*

$$\ln d_k(\Gamma_{\Phi_{\sigma_1}}^{\bar{\Phi}_{\sigma_0}}) \sim -\tau\, k^{1/2}, \qquad k \to \infty, \tag{8}$$

holds with

$$\tau = \tau(\Phi_{\sigma_1}, \bar{\Phi}_{\sigma_0}) := \left(\int_0^1 \frac{d\gamma}{(\ln F(\sigma_1, \gamma) - \ln F(\sigma_0, \gamma))^2} \right)^{-1/2}, \tag{9}$$

where the function F is defined by the formula

$$F(\sigma, \gamma) = \frac{(\sinh \sigma)^\gamma \left(\cosh \sigma + \sqrt{\gamma^2 + (\sinh \sigma)^2}\right)}{\left(\gamma \cosh \sigma + \sqrt{\gamma^2 + (\sinh \sigma)^2}\right)^\gamma} \tag{10}$$

for $0 \leqslant \gamma \leqslant 1$ and $0 < \sigma < \infty$.

The function (10) first appeared in [28] in the context of the following result about the asymptotics of the associated Legendre functions P_n^m as the ratio m/n tends to some number from the interval $[0, 1]$.

PROPOSITION 5. *The limit*

$$\lim_{\substack{n \to \infty \\ m/n \to \gamma}} \left(\frac{P_n^m(\cosh \sigma)}{n^m} \right)^{1/n} = \frac{(1+\gamma)^\gamma}{e^\gamma (1-\gamma)^{1-\gamma}}\, F(\sigma, \gamma) \tag{11}$$

exists, where $n, m \in \mathbb{Z}_+$, $m \leqslant n$, $0 \leqslant \gamma \leqslant 1$, and $0 < \sigma < \infty$.

To prove Theorem 4 we need the following uniform version of this fact, which will be proved in Section 6.

PROPOSITION 6. *For any $\varepsilon > 0$ there exists a number n_0 such that for any $n \geqslant n_0$ and any m the estimates*

$$e^{-\varepsilon} F(\sigma, m/n) \leqslant \left(\frac{P_n^m(\cosh \sigma)}{c(m, n)} \right)^{1/n} \leqslant e^{\varepsilon} F(\sigma, m/n), \tag{12}$$

hold, where

$$c(m, n) = \left(\frac{n+m}{n} \right)^m \cdot \frac{n!}{(n-m)!}. \tag{13}$$

For the oblate spheroids (3) we shall prove the following result:

THEOREM 7. *The asymptotic*

$$\ln d_k(\Gamma_{\Psi_{\sigma_1}}^{\bar{\Psi}_{\sigma_0}}) \sim -\tau k^{1/2}, \qquad k \to \infty,$$

holds with

$$\tau = \tau(\Psi_{\sigma_1}, \bar{\Psi}_{\sigma_0}) := \left(\int_0^1 \frac{d\gamma}{(\ln E(\sigma_1, \gamma) - \ln E(\sigma_0, \gamma))^2} \right)^{-1/2}, \tag{14}$$

where the function E is defined by the formula

$$E(\sigma, \gamma) = \frac{(\sinh \sigma + \sqrt{(\cosh \sigma)^2 - \gamma^2})(\cosh \sigma)^\gamma}{(\gamma \sinh \sigma + \sqrt{(\cosh \sigma)^2 - \gamma^2})^\gamma} \tag{15}$$

for $0 \leqslant \gamma \leqslant 1$ and $0 < \sigma < \infty$.

4. SPHEROIDAL HARMONIC EXPANSIONS

Here, following the well-known monograph [8], we consider some natural common orthogonal bases for the families (2) and (3).

First we deal with the family (2) of prolate spheroids. It is convenient to use the corresponding spheroidal coordinates (η, θ, φ), which are connected with the Cartesian coordinates $t = (t_1, t_2, t_3)$ as follows,

$$t_1 = \cosh \eta \cos \theta,$$

$$t_2 = \sinh \eta \sin \theta \cos \varphi,$$

$$t_3 = \sinh \eta \sin \theta \sin \varphi,$$

where $0 \leqslant \eta < \infty$, $0 \leqslant \theta \leqslant \pi$, and $-\pi < \varphi \leqslant \pi$.

Let us denote by G_σ the Hilbert space defined as the completion of the set of all complex-valued harmonic polynomials with respect to the norm

$$\|x\|_{G_\sigma} = \left(\int_{-\pi}^{\pi} d\varphi \int_0^{\pi} |\tilde{x}(\sigma, \theta, \varphi)|^2 \sin\theta \, d\theta \right)^{1/2}, \tag{16}$$

where $\tilde{x}(\eta, \theta, \varphi) = x(t)$. The system of harmonic polynomials

$$u_{m,n}(t) = \tilde{u}(\eta, \theta, \varphi) = P_n^{|m|}(\cosh\eta)\, P_n^{|m|}(\cos\theta)\, e^{im\varphi}, \tag{17}$$

with $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}$, and $-n \leqslant m \leqslant n$, forms an orthogonal basis in each Hilbert space G_σ, $0 < \sigma < \infty$. The norms of the polynomials (17) are expressed by the formula

$$\|u_{m,n}\|_{G_\sigma} = a(m, n)\, P_n^{|m|}(\cosh\sigma), \qquad 0 < \sigma < \infty, \tag{18}$$

where (cf. item 247 in [8])

$$a(m, n) = \left(\frac{4\pi\,(n + |m|)!}{(2n+1)(n - |m|)!} \right)^{1/2} \tag{19}$$

does not depend on σ.

The following fact will be useful for studying the asymptotics (8).

PROPOSITION 8. *The linear continuous imbeddings*

$$h(\bar{\Phi}_\sigma) \subset G_\sigma \subset h(\Phi_\sigma) \tag{20}$$

hold for $0 < \sigma < \infty$.

Considering the family of oblate spheroids (3) it is convenient to use the spheroidal coordinates

$$\begin{aligned} t_1 &= \sinh\eta \cos\theta, \\ t_2 &= \cosh\eta \sin\theta \cos\varphi, \\ t_3 &= \cosh\eta \sin\theta \sin\varphi \end{aligned} \tag{21}$$

with $0 \leqslant \eta < \infty$, $0 \leqslant \theta \leqslant \pi$, and $0 \leqslant \varphi < \pi$. The intersection of all Ψ_σ is the disc

$$\bar{\Psi}_0 := \{ t = (t_1, t_2, t_3) : t_1^2 + t_2^2 \leqslant 1,\ t_3 = 0 \}.$$

The formula (16), considered in coordinates (21), gives the corresponding family of Hilbert spaces H_σ, $0 < \sigma < \infty$. The system of harmonic polynomials

$$v_{m,n}(t) = \tilde{v}_{m,n}(\eta, \theta, \varepsilon) = P_n^{|m|}(i \sinh\eta)\, P_n^{|m|}(\cos\theta)\, e^{i\,m\theta}$$

with $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}$, and $-n \leqslant m \leqslant n$ forms a common orthogonal basis for Hilbert spaces H_σ, $0 < \sigma < \infty$ with the following expression for their norms,

$$\|v_{m,n}\|_{H_\sigma} = a(m,n)\, |P_n^{|m|}(i \sinh \sigma)|, \qquad 0 < \sigma < \infty,$$

where $a(m, n)$ is defined by (20).

PROPOSITION 9. *The linear continuous imbeddings*

$$h(\bar{\Psi}_\sigma) \subset H_\sigma \subset h(\Psi_\sigma)$$

hold for $0 < \sigma < \infty$.

5. ASYMPTOTICS FOR ASSOCIATED LEGENDRE FUNCTIONS

Here we are proving Proposition 6 and its analogue for oblate spheroids. To get the uniform relation (12) we use, as in [28], the following integral representation for the associated Legendre functions [8, item 63],

$$P_n^m(\cosh \sigma) = b\, (\sinh \sigma)^m \int_0^\pi (\cosh \sigma + \cos \varphi \sinh \sigma)^{n-m} (\sin \varphi)^{2m}\, d\varphi, \tag{22}$$

where

$$b = b(m, n) = \frac{2^m m!\, (n+m)!}{(2m)!\, (n-m)!}. \tag{23}$$

We again study the asymptotic behavior of the related integral

$$I = I_n(\sigma, \gamma) = \int_0^\pi A(\sigma, \gamma; \varphi)^n\, d\varphi, \tag{24}$$

where

$$A(\varphi) = A(\sigma, \gamma; \varphi) = (\cosh \sigma + \cos \varphi \sinh \sigma)^{1-\gamma} (\sin \varphi)^{2\gamma}. \tag{25}$$

However, now we apply the modified, rather roughened Laplace method [3, 4] and get only some coarse estimate of this integral instead of exact asymptotics, taking care of uniformity in our final destination (12). It is quite easy to check that the function (25) has its only maximum on the segment $[0, \pi]$, which is attained at the point $\varphi_0 \in [0, \pi/2]$, defined by the relation $(\gamma \neq 0)$

$$\cos \varphi_0 = \frac{(1-\gamma) \sinh \sigma}{\gamma \cosh \sigma + \sqrt{\gamma^2 + (\sinh \sigma)^2}};$$

if $\gamma = 0$ the maximum is attained at $\varphi_0 = 0$. Thus, we immediately get the upper estimate

$$I \leqslant \pi\, A(\varphi_0)^n. \tag{26}$$

It is easy to verify that the derivative $A'(\varphi)$ is non-positive on the segment $[\varphi_0, 3\pi/4]$ and bounded on it from below by some negative constant $-M$, independent of $\gamma \in [0, 1]$. Therefore we get the following estimate from below for the integral (24):

$$I \geqslant -\frac{1}{M} \int_{\varphi_0}^{3\pi/4} A(\varphi)^n\, A'(\varphi)\, d\varphi \geqslant \frac{A(\varphi_0)^{n+1} - A(3\pi/4)^{n+1}}{M(n+1)}. \tag{27}$$

After some calculations we get

$$(\sinh \sigma)^\gamma\, A(\varphi_0) = \frac{(2\gamma)^\gamma}{1+\gamma}\, F(\sigma, \gamma). \tag{28}$$

Finally, taking into account the estimates

$$A(\sigma, \mu, \varphi)^n\, e^{-(\mu-\lambda)\sigma n} \leqslant (\cosh \sigma + \cos \varphi \sinh \sigma)^{m-n}\, (\sin \varphi)^{2m}$$
$$\leqslant A(\sigma, \mu, \varphi)^n\, e^{(\mu-\lambda)\sigma n}$$

for $\lambda \leqslant m/n \leqslant \mu$, and the relations (18), (19), (26), (27), (28), (22), and (23), we obtain the desired uniform estimates (12).

Much as in the above considerations, we can prove

Proposition 10. *For any $\varepsilon > 0$ there exists a number n_0 such that for any $n \geqslant n_0$ and any m the estimates*

$$e^{-\varepsilon}\, E(\sigma, m/n) \leqslant \left(\frac{P_n^m(i \sinh \sigma)}{c(m, n)}\right)^{1/n} \leqslant e^{\varepsilon} E(\sigma, m/n) \tag{29}$$

hold, where the sequence $\{c(m, n)\}$ and the function E are defined by (13) *and* (15), *respectively.*

6. ASYMPTOTICS OF KOLMOGOROV DIAMETERS

6.1. *Proof of Theorem* 4. First we examine the asymptotics of Kolmogorov diameters for pairs of Hilbert spaces (16). Namely we prove

PROPOSITION 11. *Let G_α and G_β be Hilbert spaces, corresponding to a pair of prolate spheroids Φ_α and Φ_β in accordance with* (16), $0<\alpha<\beta<\infty$. *Then the asymptotics*

$$d_k(G_\beta, G_\alpha) \sim -\tau k^{12}, \qquad k \to \infty, \tag{30}$$

holds, where

$$\tau = \tau(\Phi_\beta, \bar{\Phi}_\alpha) = \left(\int_0^1 \frac{d\gamma}{(\ln F(\beta, \gamma) - \ln F(\alpha, \gamma))^2} \right)^{-1/2}. \tag{31}$$

Proof. The system (17) is a common orthogonal basis for the spaces G_α and G_β, and due to (18)

$$\frac{\|u_{m,n}\|_{G_\beta}}{\|u_{m,n}\|_{G_\alpha}} = \frac{P_n^{|m|}(\cosh \beta)}{P_n^{|m|}(\cosh \alpha)}. \tag{32}$$

Therefore, by Proposition 2, it is sufficient to study the asymptotic of the sequence $\mu_k = \mu_k(G_\alpha, G_\beta)$ (6), which coincides with the sequence obtained from the sequence (32) after it is arranged in non-decreasing order. To this end it is convenient to study asymptotic behavior of the "counting" function

$$\phi(t) := |\{k : \ln \mu_k \leqslant t\}|, \tag{33}$$

where $|A|$ means the number of elements of the finite set A. By (32) we have

$$\phi(t) = |\{(m, n), \; -n \leqslant m \leqslant n : \ln P_n^{|m|}(\cosh \beta) - \ln P_n^{|m|}(\cosh \alpha) \leqslant t\}|. \tag{34}$$

From Proposition 6 we easily get that for any $\delta>0$ there exists a number n_0 such that for $n \geqslant n_0$ and any m the following estimates are fulfilled:

$$\begin{aligned} n(\ln F(\beta, m/n) - \ln F(\alpha, m/n))(1-\delta) \\ \leqslant \ln P_n^m(\cosh \beta) - \ln P_n^m(\cosh \alpha) \\ \leqslant n(\ln F(\beta, m/n) - \ln F(\alpha, m/n))(1+\delta). \end{aligned}$$

Therefore, by (34), the following inequalities are true,

$$N(t(1+\delta)^{-1} D)-(n_0+1)^2 \leqslant \phi(t) \leqslant N(t(1-\delta)^{-1} D)+(n_0+1)^2, \quad (35)$$

where

$$D=\left\{(x, y) \in \mathbb{R}^2 : |x| \leqslant y, 0 \leqslant y \leqslant \frac{1}{\ln F(\beta, x/y)-\ln F(\alpha, x/y)}\right\}. \quad (36)$$

Since D is obviously measurable in the Jordan sense, by Lemma 3 from (35) and (36) we obtain the asymptotic

$$\phi(t) \sim \operatorname{mes}(D)\, t^2. \quad (37)$$

Quite simple calculations give us

$$\operatorname{mes}(D)=\int_0^1 \frac{d\gamma}{(\ln F(\beta, \gamma)-\ln F(\alpha, \gamma))^2}. \quad (38)$$

Finally, it is easy to check that the asymptotics (37) and (38) for the counting function (33) imply the asymptotic for the sequence μ_k

$$\ln \mu_k(G_\beta, G_\alpha) \sim \tau k^{1/2},$$

where the constant τ is defined by (31). Hence, taking into account (7), the asymptotic (30) is proved.

PROPOSITION 12. *Let $0<\sigma_0<\sigma_1<\infty$ and let X_0 and X_1 be an arbitrary pair of Banach spaces such that the linear continuous imbeddings*

$$h(\bar{\Phi}_{\sigma_i}) \subset X_i \subset h(\Phi_{\sigma_i}), \qquad i=0, 1 \quad (39)$$

take place. Then the asymptotic

$$d_k(X_1, X_0) \sim -\tau(\Phi_{\sigma_1}, \bar{\Phi}_{\sigma_0})\, k^{1/2}, \qquad k \to \infty,$$

is valid.

Proof. Fix $\varepsilon>0$. We are going to show that there exists a constant $C>0$ such that the following estimates are true:

$$\frac{1}{C} d_k(G_{\sigma_1+\varepsilon}, G_{\sigma_0-\varepsilon}) \leqslant d_k(X_1, X_0) \leqslant C\, d_k(G_{\sigma_1-\varepsilon}, G_{\sigma_0+\varepsilon}). \quad (40)$$

First we note that the canonical linear continuous imbeddings

$$h(\bar{\Phi}_\beta) \subset h(\Phi_\beta) \subset h(\bar{\Phi}_\alpha) \subset h(\Phi_\alpha)$$

are obviously true for $0<\alpha<\beta<\infty$. Taking into account these and the imbeddings (20) and (39), we conclude that the following linear continuous imbeddings hold (for small enough ε):

$$G_{\sigma_1+\varepsilon} \subset X_1 \subset G_{\sigma_1-\varepsilon} \subset G_{\sigma_0+\varepsilon} \subset X_0 \subset G_{\sigma_0-\varepsilon}.$$

Applying Lemma 1 twice we obtain the estimates (40). Using these estimates, Proposition 11, and the continuity of the characteristic $\tau(\Phi_\beta, \bar{\Phi}_\alpha)$ with respect to parameters α and β, we complete the proof of the proposition.

Now Theorem 4 occurs as a particular case of previous proposition, if we put $X_0=hC(\bar{\Phi}_{\sigma_0})$ and $X_1=hB(\Phi_{\sigma_1})$ and take into account that the imbeddings (39) for these spaces are evidently true.

6.2. *Proof of Theorem* 14. Because of the close similarity to the case of prolate spheroids, we touch on only what ought to be changed in the previous proof. Using Proposition 10 instead of Proposition 6, we first prove the natural analogue of Proposition 9 adapted for the Hilbert spaces H_σ (see Section 4) and then, using Proposition 9, we get the following

PROPOSITION 13. *Let $0<\sigma_0<\sigma_1<\infty$ and let X_0 and X_1 be an arbitrary pair of Banach spaces such that the linear continuous imbeddings*

$$h(\bar{\Psi}_{\sigma_i}) \subset X_i \subset h(\Psi_{\sigma_i}), \qquad i=0, 1$$

take place. Then the asymptotic

$$d_k(X_1, X_0) \sim -\tau\, k^{1/2}, \qquad k\to\infty,$$

is valid, where τ is described in (14) *and* (15).

Consequently Theorem 7 follows from Proposition 13, as a particular case, if we put $X_0=hC(\bar{\Psi}_{\sigma_0})$ and $X_1=hB(\Psi_{\sigma_1})$ in it.

7. FINAL REMARKS

7.1. Using the results of Mityagin [13] and Levin and Tikhomirov [12] about the connection between asymptotics for Kolmogorov diameters $d_k(A)$ and for ε-entropy $\mathscr{H}_\varepsilon(A)$ [20, 21], we get from Theorem 8 and Theorem 11, as a corollary, the following

THEOREM 14. *Let a pair* (K, D) *be* $(\bar{\Phi}_{\sigma_0}, \Phi_{\sigma_1})$ *or* $(\bar{\Psi}_{\sigma_0}, \Psi_{\sigma_1})$. *Then the asymptotic*

$$\mathscr{H}_\varepsilon(\Gamma_D^K) \sim \frac{2}{3\tau^2} (\ln 1/\varepsilon)^3, \qquad \varepsilon \to 0. \tag{41}$$

holds with τ *as in* (9) *or as in* (14), *respectively.*

Let us note that in the case of real-valued harmonic functions the asymptotic (41) remains valid if the right-hand-side is halved, whereas the asymptotic (1) is the same in both the cases.

Problem 1. Does the limit

$$\tau = \tau(K, D) = \lim_{k \to \infty} -\frac{\ln d_k(\Gamma_D^K)}{k^{1/2}} \tag{42}$$

exist for a more or less general compact set K and its open neighborhood D?

As mentioned above, the positive answer immediately implies the asymptotic (41) with the constant τ claimed in (42).

7.2. Let K be a compact set in $\mathbb{C}$, D its open neighborhood, $H^\infty(D)$ the Banach space of all bounded analytic functions in D with the uniform norm, and A_D^K its unit ball considered as a compact subset of the space $C(K)$. Under some very non-restrictive conditions on K and D many authors gave at least four principally different proofs of the well-known Kolmogorov hypothesis about the following asymptotics (A. Vitushkin [22], K. Babenko [1], V. Erokhin [5, 6], V. Zahariuta [24], A. Levin and V. Tikhomirov [12], H. Widom [23], V. T. Nguyen [15], V. Zahariuta and N. Skiba [29], S. Fisher and Ch. Micchelli [7], *et al.*):

$$\ln d_k(A_D^K) = \ln d_k(H^\infty(D), C(K)) \sim -\frac{k}{c(K, D)}, \qquad k \to \infty,$$

$$\mathscr{H}_\varepsilon(A_D^K) \sim c(K, D)\, (\ln 1/\varepsilon)^2, \qquad \varepsilon \to 0,$$

where $c(K, D)$ is the Green capacity of the set K with respect to the domain D [2, 11] (or, shortly, the capacity of the condenser (K, D)).

For harmonic functions in $\mathbb{R}^2 = \mathbb{C}$ the asymptotics (1), though less investigated, is also known under quite general conditions on K and D [16, 17, 19, 30]; therefore, the constant τ has again a natural potential theory meaning, namely

$$\tau = \tau(K, D) = \frac{1}{2c(K, D)}.$$

However, the two-dimensional case comes as no surprise because of the close relationship between analytic and harmonic functions.

The notion of *Lh*-potential, introduced in [26, 28], proved to be an appropriate substitution for the Green potential for the studying of Hadamard-type inequalities for the moduli of harmonic functions (instead of analytic functions).

Problem 2. *How to define an adequate Lh-capacity of a condenser* (K, D), *fitted for the asymptotics* (1) *and* (41)?

On the other hand, we can consider the relations

$$c^{+}(K, D) := \limsup_{k\to\infty} -\frac{k^{1/2}}{\ln d_k(\Gamma_D^K)}; \qquad c^{-}(K, D) := \liminf_{k\to\infty} -\frac{k^{1/2}}{\ln d_k(\Gamma_D^K)}$$

as the definitions of some capacity-like characteristics for an arbitrary pair (K, D). So, we can now ask what is the meaning of such characteristics in the context of "*Lh*-potential theory" [26, 28]?

7.3. It seems that our results can be extended to an arbitrary pair of confocal ellipsoids; the asymptotics for Lame functions E_n^m (cf. item 280 in [8]), generalizing (12) and (32), are all that we need for this purpose.

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